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LETTER TO THE EDITOR

An exact functional relation for the partition function of a 2D Ising model with magnetic field

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Abstract. An exact functional relation is established for the partition function of the Ising model with magnetic field formulated on a Kagome lattice of arbitrary size N. An exact solution found in a previous work is a particular case of the result presented in this letter.

In a previous letter (Giacomini 1988, hereafter referred to as I), an exact solution for the Ising model with magnetic field formulated on the Kagome lattice (KL) was found. This solution is valid when an appropriate relation between the three interaction parameters K_1, K_2, K_3 of the model and the magnetic field H is satisfied. In that case, the partition function on the KL is related to the partition function on the honeycomb lattice (HL) without magnetic field. However, the partition function on the HL with zero magnetic field is also related to the partition function on the KL with $H=0$ (Syozi 1972). Therefore the result presented in I can be expressed as

$$Z_K(K_1, K_2, K_3, H) = A^N Z_K(K_1', K_2', K_3', H'=0) \qquad (1)$$

where the parameters K_i' are functions of K_i and H, A is a constant factor and Z_K is the KL partition function. This relation is valid when equation (15) of I is satisfied.

When expressed in this form, it is very appealing to consider the exact solution of I as a particular case of the more general relation

$$Z_K(K_1, K_2, K_3, H) = A^N Z_K(K_1', K_2', K_3', H') \qquad (2)$$

where now H' is also a function of K_i and H, and the condition $H'=0$ would be equivalent to equation (15) of I.

In this letter it is shown that this is indeed the case. For simplicity only the isotropic case $K_1 = K_2 = K_3 = K$ is analysed.

Let us consider an isotropic Ising model with interaction parameter K and magnetic field H, formulated on a KL with N sites and periodic boundary conditions. As is well known, this model is related to the HL Ising model with interaction parameter R and magnetic field $3f$ by the following expression (Naya 1954, Syozi 1972):

$$Z_{\text{honey}}(R, 3f) = P^N Z_K(K, H) \qquad (3)$$

where

$$\exp(4R) = \cosh(2L+H)\cosh(2L-H)[\cosh(H)]^{-2} \qquad (4a)$$

$$\exp(4f) = \cosh(2L+H)[\cosh(2L-H)]^{-2} \qquad (4b)$$

$$\cosh(2L) = \tfrac{1}{2}[\exp(4K)+1] \qquad (4c)$$

$$P = \tfrac{1}{2}[\exp(2K)(\exp(4K)+3)]^{1/3}[\cosh(2L+H)\cosh(2L-H)\cosh^2(H)]^{-1/4}. \qquad (4d)$$

A trivial symmetry property of the Ising model with a magnetic field h on an arbitrary lattice with an even number of sites is that the partition function remains invariant when h is transformed in $h+\mathrm{i}\pi$, where i is the imaginary unity. However, when this symmetry is applied to the magnetic field $3f$ of the HL, a surprising result is obtained. In order to change $3f$ to $3f+\mathrm{i}\pi$ it must transform f to $f+\mathrm{i}\pi/3$. But equation (4b) changes in a non-trivial fashion under such transformation, namely an additional factor $\exp(4\mathrm{i}\pi/3)$ appears. Therefore, if two different Ising models on the KL have parameters K, H and K', H', respectively, but in such a way that the two corresponding models on the HL have parameters R, $3f$ and R, $3f+\mathrm{i}\pi$, it is possible to find a relation between the two models on the KL. Taking into account (4a) and (4b), the parameters K' and H' must satisfy the following conditions:

$$\cosh(2L'+H')\cosh(2L'-H')[\cosh(H')]^{-2}=\cosh(2L+H)\cosh(2L-H)[\cosh(H)]^{-2} \tag{5a}$$

$$\cosh(2L'+H')[\cosh(2L'-H')]^{-1}=\exp(4\mathrm{i}\pi/3)\cosh(2L+H)[\cosh(2L-H)]^{-1} \tag{5b}$$

where

$$\cosh(2L')=\tfrac{1}{2}[\exp(4K')+1]. \tag{5c}$$

Therefore, it is evident that, if equations (5) are satisfied, the following relation holds:

$$P^N Z_{\mathrm{K}}(K,H)=P'^N Z_{\mathrm{K}}(K',H') \tag{6}$$

where P' is given by (4d) with K and H replaced by K' and H', respectively.

By using equations (4d), (5) and (6), explicit expressions for K' and H' as functions of K, H and for P'/P are found. The results are as follows:

$$Z_{\mathrm{K}}(K,H)=A^N Z_{\mathrm{K}}(K',H') \tag{7a}$$

$$\exp(4K')=2[(1+\alpha)/(1-\beta^2)]^{1/2}-1 \tag{7b}$$

$$\tanh(H')=\beta[(1+\alpha)/(\beta^2+\alpha)]^{1/2} \tag{7c}$$

$$A=\left(\frac{\exp(2K')[\exp(4K')+3]}{\exp(2K)[\exp(4K)+3]}\right)^{1/3}\left(\frac{\cosh(H)}{\cosh(H')}\right) \tag{7d}$$

where

$$\alpha=[\exp(8K)+2\exp(4K)-3]/4\cosh^2(H) \tag{7e}$$

$$\beta=[a+b\tanh(H)]/[1+ab\tanh(H)] \tag{7f}$$

$$b=\{[\exp(4K)+1]^2-4\}^{1/2}[\exp(4K)+1] \tag{7g}$$

$$a=-\mathrm{i}\sqrt{3}\,\mathrm{sgn}(H) \tag{7h}$$

with $\mathrm{sgn}(H)=1$ if $H=0$.

Hence, a highly non-trivial functional relation for the KL Ising model with magnetic field has been obtained from a very simple symmetry property of the same model on the HL. To the author's knowledge, this is the first time that a relation of this nature has been found for the Ising model with magnetic field.

In particular, putting $H'=0$ in (7) we have an exactly soluble case for the model with parameters K and H. Obviously, a relation between these parameters results when the condition $H'=0$ is imposed, which is as follows:

$$\tanh^2(H)=3[\tanh^2(2K)-2\tanh(2K)]^{-1} \tag{8}$$

i.e. the same equation that has been found in I (for the isotropic case) for assuring the exact solubility of the model.

As is evident from the procedure employed in its derivation, the functional relation (7) is an involutive transformation, i.e. when applied twice the initial model is obtained.

The results presented in this letter can be generalised to the anisotropic case with three interaction parameters K_1, K_2, K_3. This point, as well as the study of the analytical consequences of (7), are now under consideration.

Finally, let us emphasise that it would be very interesting to investigate this functional relation (for the anisotropic case) combined with the inversion relation (for a review see Maillard (1985)) and the disorder solution that exists for this model (Jaekel and Maillard 1985). The combined analysis of these exact results can throw a new light on the mathematical structure of the partition function of the two-dimensional Ising model with magnetic field.

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